

# Random Matrix Theory in Lattice Statistical Mechanics

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## Abstract

In this short note we collect together known results on the use of Random Matrix Theory in lattice statistical mechanics. The purpose here is two fold. Firstly the RMT analysis provides an intrinsic characterization of integrability, and secondly it appears to be an effective tool to find new integrable models. Various examples from quantum and classical statistical mechanics are presented.

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## 1 Introduction

The Random Matrix Theory (RMT) was introduced in the early fifties by E.P. Wigner Ref.(1) to study heavy nucleus. The key idea was to replace Complexity by Randomness, arguing that the Hamiltonian of a real heavy nucleus is so complicated that its full determination is out of reach. Instead of trying to include all the physical ingredients, one considers, in the RMT analysis, that the resulting operator can be seen as the representative of a suitable statistical ensemble. Of course this is only an approximation but the RMT analysis turned out to give very good results in describing many situations in nuclear physics.

After this pioneering work, the RMT analysis has been applied to many fields of physics, and also of pure mathematics. Schematically one can draw a crude analogy with the large number law: the distribution of the sum of a large number of independent random variables is, under some restrictive conditions, a Gaussian; in the same way the spectrum of a sufficiently “generic” Hamiltonian is well approximated by the “average” spectrum of operator statistical ensemble. A general presentation of the Random Matrix Theory can be found in Refs.(2; 3).

This statistical ensemble depends on the symmetries of the physical system under consideration. Actually this RMT analysis does not apply to a single Hamiltonian, but to a family of Hamiltonians, i.e. an Hamiltonian *depending on some parameters*. Four statistical ensembles are sufficient to describe the main situations one can encounter. If the family of the operator can be expressed in a basis *independent of the parameters* where all the entries of a symmetric matrix are real, then the probability distribution should be invariant under any orthogonal transformation. If one also requires independence of the entries one is led to the so-called Gaussian Orthogonal Ensemble (GOE), which is the set of symmetric matrices with entries drawn from a centered, and normalized, Gaussian distribution (except diagonal entries for which the root-mean square is two). A family of Hamiltonian is time-reversal invariant if there exists an operator  $T$  such that  $Te^{iH(\{\lambda\})t}T = ae^{-iH(\{\lambda\})t}$ . This condition is fulfilled iff there exists a unitary operator  $K$  such that  $H(\{\lambda\})K = K\overline{H(\{\lambda\})}$ , where  $K$  can be either symmetric or antisymmetric and  $\overline{H}$  denotes the conjugate. Note that any symmetric and unitary operator  $K$  can be written as the product of a unitary operator  $U$  and its transpose, namely  $K = U\tilde{U}$ , and thus one can perform a change of basis bringing the hermitian Hamiltonian  $H$  into a symmetric hermitian, and *thus real*, matrix:  $U^{-1}HU$ . One sees that if an operator is time reversal invariant, and if  $K$  is symmetric, then the GOE will apply. By contrast if  $K$  is antisymmetric, the GOE will not apply. Instead the so-called Gaussian Symplectic Ensemble (GSE) will apply. This is the ensemble of quaternion hermitian matrices. In the case where the family of the operator is not time reversal invariant, then the Gaussian Unitary Ensemble (GUE) will apply. The probability distribution is then invariant under any unitary transformation. This is the ensemble of hermitian matrices with both real and imaginary parts of each entries being independent and drawn from a Gaussian distribution. The fourth case is precisely the very peculiar case of *integrable* models. In this case there exists a basis *independent of the parameters* in which the Hamiltonian is diagonal since there are as many commuting operators as the size of the Hilbert space. The ensemble to introduce here is simply the Random Diagonal matrix Ensemble (RDE), i.e. diagonal matrices with random independent diagonal entries.

In the next section we sketch how to apply these ideas to quantum and classical lattice statistical mechanics. In the last section we illustrate the RMT with various models of lattice statistical mechanics.

## 2 Application to lattice statistical mechanics

When applied to quantum statistical mechanics, it is clear that RMT analysis has to be performed on the Hamiltonian itself. However for classical statistical mechanics, it is not clear what is the operator to be considered. Take for

example the classical Ising model. The possible values of the energy are  $E_k = -(N_e - 2k)J$ , where  $N_e$  is the number of edges of the lattice and  $J$  is the coupling constant: the spectrum is totally rigid and thus will not be described properly by any of the four statistical ensembles introduced in the previous section. It has been shown that the proper operator to consider for classical models is the *transfer matrix* Ref.(4). Actually the procedures we present below always apply either to the quantum Hamiltonian for the quantum model or to the transfer matrix for the classical model.

Before performing the RMT analysis on a given family of Hamiltonian one has to consider its symmetries. By symmetry we mean a linear operator *independent of the parameters* acting on the same Hilbert space and which commutes with the family of Hamiltonian. The set of such operators forms a group. Using the irreducible representations of this group one can find a basis in which the Hamiltonian is block-diagonal, each block defining a sector indexed by quantum numbers. Obviously, states belonging to different sectors are not correlated and the analysis has to be performed separately in each sector. These symmetries are usually the lattice symmetries (i.e. the automorphy group of the lattice in graph theory language), the spin symmetries (for example the  $O(3)$  spin and pseudo-spin symmetry of the Hubbard model Ref. (5)), and the color symmetry (for example the permutation of the states in a  $q$ -state Potts model). The number of those symmetries is a power of the number of sites of the lattice whereas the Hilbert space size grows exponentially with this number. Consequently the size of each block remains, after the block diagonalization, an exponential function of the number of sites of the lattice. However in the very special case of an integrable family of Hamiltonian, the number of symmetries equals the Hilbert space size, and a total reduction would lead to a completely diagonal matrix. In practice one does not know all the symmetries and the block diagonalization is only partial, leading to blocks well represented by RDE.

The density of states of the various models of lattice statistical mechanics are very different of each other. Obviously no universality can be found in the raw spectrum. Instead one can write the integrated density of states as  $\rho(\lambda) \simeq \text{regular}(\lambda) + \text{scale} \times \text{universal}(\lambda)$ , where the regular part does depend on the model while the universal part does not. The possible forms of the universal part are given by the four ensembles described in the first section. The procedure to extract this universal part is known as the *unfolding* of the spectrum. It has been described in many references, and amounts to transforming the raw eigenvalues into unfolded eigenvalues, which have a local density of states very close to one everywhere in the spectrum.

Once the spectrum has been sorted according to quantum numbers and properly unfolded, it remains to “compare” it with the spectrum of the four ensembles GOE, GUE, GSE or RDE. For a given Hamiltonian the eigenvalues

are well determined and the joint probability distribution of the eigenvalues is simply a Dirac measure. It will never be the joint probability distribution of the eigenvalues of the Gaussian Ensembles which is well known to be

$$P_\beta(\lambda_1, \dots, \lambda_n) = C \prod_{i < j} |\lambda_i - \lambda_j|^\beta \exp\left(-A \sum_i \lambda_i^2\right)$$

where  $\beta = 1, 2, 4$  respectively for GOE, GUE and GSE is the level repulsion. Instead, one can restore “probabilistic” properties introducing the level spacings. Sorting the unfolded eigenvalues in ascending order, the set of the differences  $s = \lambda_i - \lambda_{i-1}$  between consecutive eigenvalues does form a distribution which can be compared to the four reference level spacings:

$$\begin{aligned} P_{\text{RDE}}(s) &= \exp(-s) & P_{\text{GOE}}(s) &= \frac{\pi}{2} \exp(-\pi s^2/4) \\ P_{\text{GUE}}(s) &= \frac{2^5}{\pi^2} \exp(-4s^2/\pi) & P_{\text{GSE}}(s) &= \frac{64^3}{9^3 \pi^3} s^4 \exp(-64s^2/9\pi) \end{aligned}$$

Note that the above expression are only approximations of the corresponding level spacing distribution<sup>1</sup>. In practice it is useful to use a parametrized distribution which extrapolate between RDE and GOE. Using the following distribution

$$P_\beta(s) = c(\beta + 1)s^\beta \exp(-cs^{\beta+1}) \tag{1}$$

one can find the value of  $\beta$  realizing the best fit: a small value of  $\beta \sim 0.1$  will indicate an integrable model, whereas a value of  $\beta \sim 0.9$  will indicate a GOE statistic of the eigenvalues and consequently a time-reversal model.

If one want to test how close the given Hamiltonian is from the statistical ensemble, one compute other quantities involving more than only two consecutive eigenvalues. One of these quantity is the so-called rigidity

$$\Delta_3(L) = \left\langle \frac{1}{L} \min_{a,b} \int_{\alpha-L/2}^{\alpha+L/2} (\rho(\lambda) - a\lambda - b)^2 \right\rangle_\alpha$$

here the bracket an average over all the possible position of the “window” of width  $L$ . The behavior of the rigidity for the RDE, GOE, GUE and GSE is known and is presented for comparison in the figures of the next section.

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<sup>1</sup> The actual distributions are in fact related to Painlevé transcendents.

### 3 Examples

This section is devoted to various examples. We will see that indeed non integrable model compare extremely well with the corresponding Gaussian ensemble, and also that the spectra of integrable models are, in many respect, close to a set of independent numbers (RDE) Refs(6; 7; 8; 9; 10; 11; 12; 13; 5; 14; 15).

#### 3.1 The Generalized Hubbard Chain.

We begin with the generalized Hubbard Chain (see Ref.(8)) which describes a set of electrons (or any spin one-half particles) on a chain and interacting via both a Coulomb repulsion  $U$ , a proximity interaction  $V$  and a Heisenberg coupling  $J$ . The results presented in this section originate in a long-standing collaboration of H. Meyer with the authors (see Ref. (8)). The Hamiltonian reads:

$$H = t \sum_{i,\sigma} c_{i+1,\sigma}^\dagger c_{i,\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} + V \sum_i n_i n_{i+1} + J \sum_i \vec{S}_i \cdot \vec{S}_{i+1} \quad (2)$$

This model is particularly interesting since it may, or may not, be integrable, depending on the parameters. In this context integrable means that the eigenfunctions actually have the form proposed in the Bethe ansatz (see Ref.(20)), or in its refined nested form. The known integrable cases are summarized in the following table:

	$U/t$	$V/t$	$J/t$
Hubbard	$\forall$	0	0
$t - J$ supersymmetric	$\infty$	$\pm 1/2$	$\mp 2$
	$\infty$	$\pm 3/2$	$\pm 2$
t-0	$\infty$	0	0
$XXZ$ chain	$\infty$	$\forall$	0

In Fig. 1, we compare the level spacing and the rigidity in two paradigm cases Ref.(8). One case  $t = 1$ ,  $U = 10$  and  $J = V = 0$  corresponds to an integrable case, and, indeed, the level spacing  $P(s)$  and the rigidity  $\Delta_3$  are in good agreement with the prediction of independent eigenvalue (RDE), whereas the second case  $t = U = 1$ ,  $V = 0$  and  $J = 2$  corresponds to a generic non integrable case and is in good agreement with the GOE ensemble. The

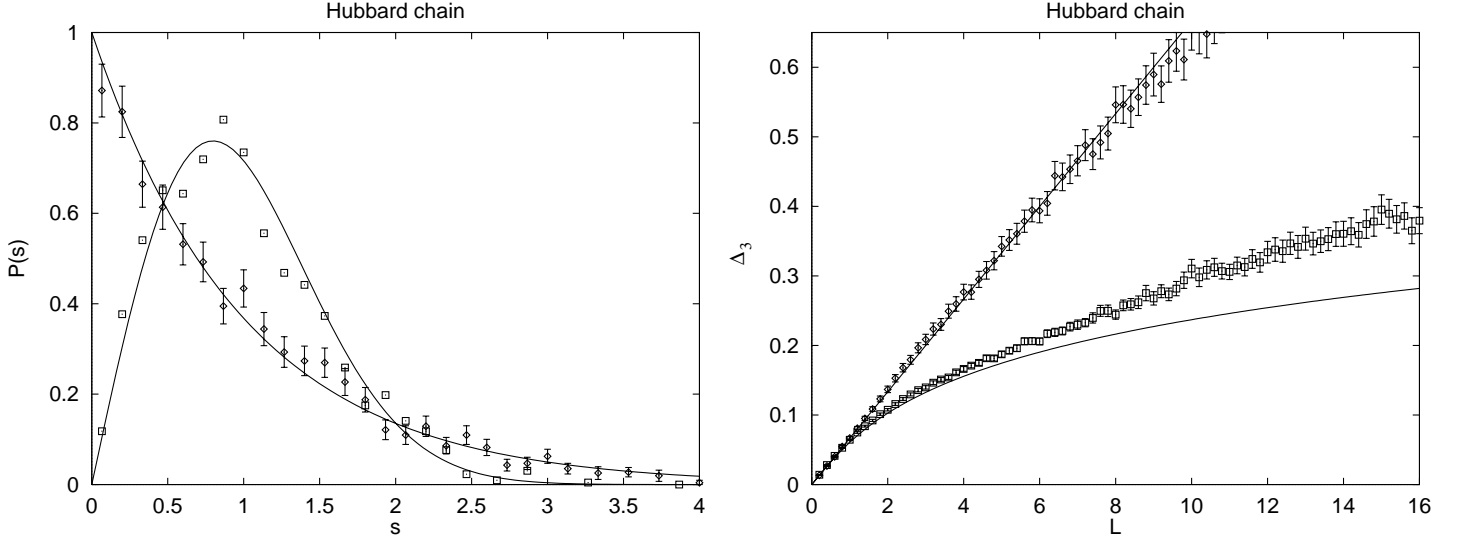


Figure 1. Generalized Hubbard Chain

Hamiltonian Eq.2 being real, one expects GOE rather than GUE or GSE.

To better follow how the level repulsion  $\beta$  behaves in the different region of the parameter space, we define a path in the phase space, and record  $\beta$  as we move along this path. To be specific we simply fixed  $U = 0$  and, for different values of  $V$ , we vary  $J$ . The parameter  $\beta$  corresponds to a best fit of the distribution Eq. 1. On figure Fig. 2 the specific integrable points are clearly seen as points for which the parameter  $\beta$  drops to zero, in excellent agreement with the previous table.

### 3.2 The Chiral Quantum Potts Chain.

We now turn to another quantum Hamiltonian : the quantum chiral Potts chain Refs.(16; 17). The corresponding transfer matrix has a *higher genus* integrability Refs. (18; 19). So it is natural to wonder if also the Gaussian ensembles provide correct descriptions of the spectrum. The quantum Hamiltonian reads:

$$H = \sum_j \sum_{n=1}^{N-1} \left[ \overline{\alpha}_n (X_j)^n + \alpha_n (Z_j Z_{j+1}^\dagger)^n \right] \quad (3)$$

where  $X_j = I \otimes \cdots \otimes X \otimes \cdots I$  and  $Z_j = I \otimes \cdots \otimes Z \otimes \cdots I$  operators  $X$  and  $Z$  are in position position  $j$ ,  $I$  is the unit  $q \times q$  matrix,  $X_{ij} = \delta_{i,j+1 \bmod(N)}$

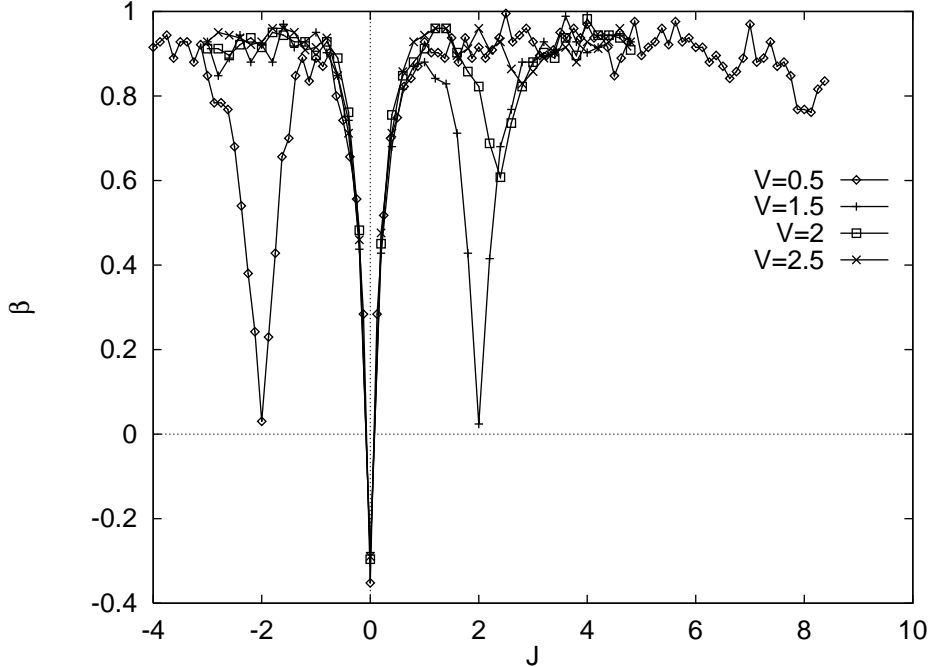


Figure 2. Repulsion as a function of the coupling (see text).

and  $Z_{ij} = \delta_{i,j} \exp(2\pi i(j-1)/N)$ . Moreover Hamiltonian Eq. 3 is complex, at least for general values of the parameters, and so one expects, *a priori*, a GUE statistic. An integrability condition has been found for this model Ref.(18; 19). Restricting ourself to values of the parameters which ensure that Hamiltonian Eq. 3 is hermitian, we have performed a RMT analysis which allows us to conclude that i) along the integrable variety the RDE is an adequate description and ii) for generic point the GOE is the correct description. Point ii) is quite surprising, since the GUE was expected. This means that there exists a basis, independent of the parameters, in which the Hamiltonian is *real*. We have been able to find this basis for sizes smaller than  $L = 6$ . Note that this property implies the existence of a unitary operator  $K$  which is extremely over-constrained (see the introduction). From our numerical results, we conjecture the existence of such a basis for any chain size  $L$ .

### 3.3 The three-dimensional Ising model.

The three-dimensional Ising model is certainly one of the most challenging model of lattice statistical physics. In particular the properties of the critical point are debated. To clarify this question we have performed a RMT analysis, see Ref.(9) . We start with an anisotropic Ising model on a cubic lattice. In two directions the couplings have the value  $K_2$  while, in the third, it has the value  $K_1$ . When  $K_1 = K_2$  this is the usual isotropic cubic Ising model, and when  $K_1 = 0$  it reduces to the isotropic two dimensional square lattice. We

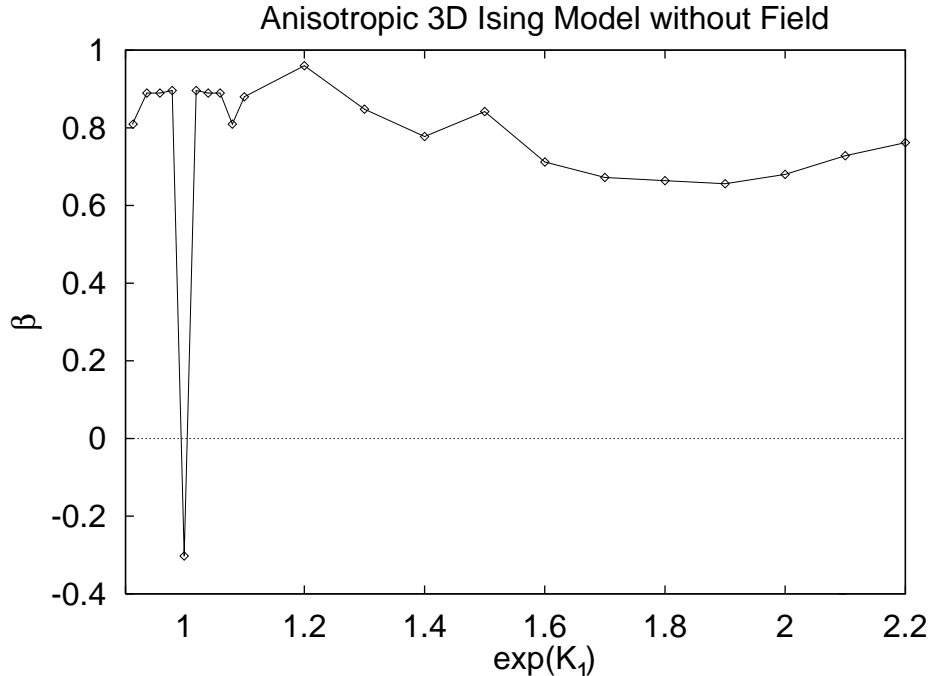


Figure 3. Three-dimensional Ising model

keep constant  $K_2 = 1$  and vary  $K_1$  in a range starting from a small negative coupling constant value to a value sufficiently large to be certainly larger than the critical value which can be crudely evaluated by different means. The results are summarized in Figure 3. It is clear, from this figure, that the critical point *does not* show any trace of a possible *integrability-like property*<sup>2</sup>. This is in contrast with the results in the proximity of the two dimensional case, where the absence of level repulsion is clearly seen.

### 3.4 The Ising model on the Kagomé lattice.

To conclude this short note we would like to mention that we have applied the RMT analysis to the Ising model on the Kagomé lattice. F.Y. Wu, pointed out to us that it would be interesting to test the critical point of the Kagomé lattice with the RMT analysis. The critical manifold is not known, but F.Y. Wu conjectured some algebraic variety for the anisotropic model see Ref. (21). From this RMT analysis, a value for the critical temperature of the isotropic model can be deduced. This value is very close to the one obtained from Monte-Carlo simulations. Fig. 4 (taken from Ref.(10)) presents the level spacing distribution for the conjectured critical value  $K_W$ , as well as for a generic value  $K = 2$ . It confirms a good agreement with the conjectured value, but,

<sup>2</sup> Let us recall that the reduction, in some scaling limit, of the *critical* three-dimensional model to some (integrable) conformal field theory, thus yielding rational exponents, had been suggested by several authors.



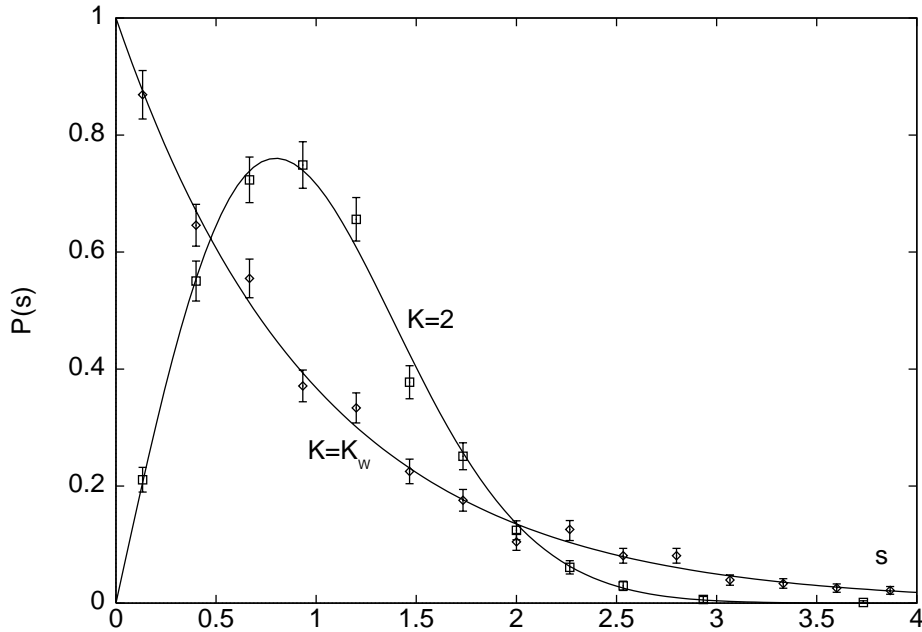


Figure 4. Ising model on a Kagomé lattice.

mainly, it shows that the *critical point is integrable*, in contrast to the example of the three dimensional critical point.

## 4 Conclusions

We have seen that RMT analysis could provide an alternative approach to integrability and, to some extent, an alternative definition to Bethe integrability or to Yang-Baxter integrability. It also gives an operational way of testing integrability. We have found a time-reversal-like unexpected symmetry in the Chiral Quantum Potts chain for which higher genus integrability occurs, the spectrum being correctly described by the RDE. Many other classical spin or vertex models, as well as various quantum models, have also been investigated, all leading to the same conclusions developed in this note. Furthermore we have shown that the three-dimensional Ising model *does not* have this property of independent eigenvalues for the spectrum of the transfer matrices. This strongly suggests that this model will not be solved without a genuinely new method, even at criticality.

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